이학박사 학위논문

Global Quantum Discord derived from Tsallis entropy in Multipartite systems

(살리스 엔트로피로 일반화 한 글로벌 양자 discord)

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지도교수 김 도 한

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위 원 장 천 정 희 (인)

부위원장 김 도 한 (인)

위 원 지 동 표 (인)

위 원 이 수 준 (인)

위 원 Yasuhiro Takahashi(인)

Global Quantum Discord derived from Tsallis entropy in Multipartite systems

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Abstract

Global Quantum Discord derived from Tsallis entropy in Multipartite systems

Kyungjin Lee
Department of Mathematical Sciences
The Graduate School
Seoul National University

In this dissertation, we introduce the generalized global quantum discord, called the q-global quantum discord (q-GQD), which is defined in terms of the quantum Tsallis entropy, and obtain the analytical expressions of q-GQD for the special classe of multi-qubit states. Furthermore, we show a monogamy inequality for pairwise quantum correlation, which implies that the sum of pairwise quantum correlations is upper bounded by multipartite quantum correlation with respect to q-GQD.

Keywords: quantum discord, Global quantum discord, Tsallis entropy, monogamy
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Chapter 1

Introduction

In quantum information theory, the research on how to quantify and to measure the quantum correlations in multipartite systems are still meaningful challenges and the most crucial open questions [1, 2, 3, 4]. The study on characterization of quantum correlations has been progressed during the last two decades. In bipartite systems with low dimension Werner has introduced the result based on the entanglement-separability dichotomy [5] which has been very significant the framework of paradigm. In the framework of this approach it has become obvious that the correlation in a quantum state can be classified as either classical or quantum. However some results have shown that there exist separable quantum states with correlation which can perform some useful quantum tasks not simulated by classical methods [6, 7, 8, 9, 10, 11, 12]. Therefore, in recent one decade it has been introduced that there exists an important quantity to represent quantum correlation, called the *quantum discord*, which are different from previous entanglement measures [13, 14, 16, 17].

In this viewpoint as the first attempt, Ollivier and Zurek introduced the original definition of the quantum discord of bipartite states over projective measurements [14] which was generalized over general measurements rank-1 general measurements or Neumark extensions by Jianwei Xu [19]. It was furthermore found that there exists the difference between two natural expressions about quantum analogue of the classical mutual information. Recently, Okrasa and Walczak provided the quantum discord in multipartite systems[1] as the minimal amount of the difference which is generated when a local quantum operation on one system is performed [20].

A generalization of quantum discord to global quantum system was suggested by Rulli and Sarandy [18]. They defined a global measure of quantum discord obtained by a systematic extension of the bipartite and multipartite systems called the *global quantum discord* (GQD). Jianwei Xu then proposed an equivalent expression for GQD, and obtained the analytical expressions of GQD for two classes of multi-qubit states [22]. Majtey *et al.* and Jurkowski had proposed the quantum discord in terms of the quantum Tsallis entropy instead of the von Neumann entropy [23, 24].

In Chapter 3 we describe in detail the aforementioned results which have studied with regard to generalization of quantum discord. In Chapter 4 we introduce the generalized global quantum discord, called the *q-global quantum discord* (q-GQD), which is defined in terms of the quantum Tsallis entropy, and provide an equivalent expression for q-GQD. Furthermore we obtain the analytical expressions of q-GQD for the special classes of multi-qubit states and also prove the nonnegativity of q-GQD for the classes. In particular we find that this q-GQD has an unusual feature related to order and also show a monogamy inequality for pairwise quantum correlations which implies that the sum of pairwise quantum correlations is upper limited by the multipartite quantum correlation with respect to q-GQD.

[1]Other definitions of quantum discord to multipartite states have been considered in difference scenarios [21].

Chapter 2

Preliminaries

2.1 Axioms of quantum mechanics

Quantum theory is usually approached with a mathematical description which is demonstrated through a lot of experiments for a century. We first introduce several mathematical axioms for describing quantum states, observables and measurements.

Axiom 1 (State). *A quantum state represents a proper characteristic of physical system such as energy levels of an electron, polarizations of a photon, and so on. Mathematically a quantum state in a closed system is denoted by a ray in a Hilbert space over the complex field $\mathbb{C}$ which is an equivalence class of vectors identical up to the scalar multiplication (or the global phase) that is*

$$|\psi\rangle \equiv e^{i\alpha}|\psi\rangle$$

for any complex number α.

A quantum state denoted by a vector as mentioned above is called a *pure* state. Especially a quantum state in two-dimensional Hilbert space is called a *qubit* and the above two states actually compose a canonical basis for a qubit system. Moreover, we also use its dual form which is a linear operator transforming a vector to a scalar and denoted as follows

$$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad \text{and} \quad \langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Axiom 2 (Observables)**.** *An observable A is a self-adjoint (or Hermitian) operator on a complex Hilbert space, that is, $A = A^\dagger$, where the adjoint operator $A^\dagger$ of A is uniquely determined by the condition that for all vectors $|\phi\rangle$ and $|\psi\rangle$*

$$\langle \phi | A\psi \rangle = \left\langle A^\dagger \phi | \psi \right\rangle,$$

and is specified by the composition of transpose and complex conjugation of A under the canonical basis.

If A and B are observables, then $A + B$, $AB + BA$, and $i(AB - BA)$ are always so, but, AB is self-adjoint if and only if A and B commute. Observables have a good property that all eigenvalues of them are real numbers and moreover eigenvectors corresponding to different eigenvalues is orthogonal to each other, so that an observable A can be expressed as

$$A = \sum_n \lambda_n P_n.$$

Here each λ_n ($\lambda_i \neq \lambda_j$ if $i \neq j$) is real eigenvalue of A and P_n is the orthogonal projection onto the subspace spanned by all eigenvectors corresponding to λ_n. If λ_n is non-degenerate, then P_n is an one-dimensional projection and denoted by $P_n = |n\rangle\langle n|$. The all P_n's satisfy that

$$P_n P_m = \delta_{n,m} P_n \quad \text{and}$$

$$P_n = P_n^\dagger.$$

The outcomes of measurement result in the eigenvalues of the above observables, which occur with certain probabilities as follows.

Axiom 3 (Measurement)**.** *Suppose that we measures a quantum state $|\psi\rangle$ with an observable*

$$A = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|,$$

where λ_k are usually different and $|\psi_k\rangle$ are mutually orthogonal. After the measurement, the outcome λ_k is obtained with probability

$$Prob(\lambda_k) = \| \langle\psi_k|\psi\rangle \|^2 = \langle\psi|\psi_k\rangle\langle\psi_k|\psi\rangle,$$

and the given state $|\psi\rangle$ is collapsed to the corresponding eigenvector $|\psi_k\rangle$.

The expectation value of the outcomes is therefore

$$\begin{aligned} E_A(|\psi\rangle) &= \sum_k \lambda_k Prob(\lambda_k) && (2.1)\\ &= \sum_k \lambda_k \langle\psi|\psi_k\rangle\langle\psi_k|\psi\rangle && (2.2)\\ &= \mathrm{tr}(\sum_k \lambda_k |\psi_k\rangle\langle\psi_k||\psi\rangle\langle\psi|) && (2.3)\\ &= \mathrm{tr}(A|\psi\rangle\langle\psi|). && (2.4) \end{aligned}$$

Right after the measurement, if we repeat immediately the same measurement on the resulting state, then the same outcome is obtained again with a certainty as in the general thought.

2.2 Density operators

In general, most of quantum systems are not closed except for the whole universe. It is actually very difficult to maintain a quantum system in the pure state, because of quantum decoherence caused by the interaction with its environmental system such as quantum interference.

Let us consider a bipartite quantum system consisting of two qubits which is indexed by A and B, and then observe only one part of them. Using the tensor product, any bipartite state can be denoted by

$$|\Psi\rangle_{AB} = \sum_{ij} \alpha_{ij} |i\rangle_A \otimes |j\rangle_B,$$

where $\sum_{ij} |a_{ij}|^2 = 1$. Hereafter we will sometimes omit the tensor notation for simplicity.

Suppose that we performs a measurement only on subsystem A with observable $M_A \otimes I = \{M_A(\alpha) \otimes I\}$. By the Axiom 3, the expectation value of measurement outcomes are exactly

$$\begin{aligned}
\mathrm{tr}((M_A(\alpha) \otimes I)|\Psi\rangle_{AB}\langle\Psi|) &= \mathrm{tr}_A\left(\sum_{ijkl} \delta_{jl}\alpha_{ij}\overline{\alpha_{kl}}(M_A(\alpha)|i\rangle_A\langle k|)\right) &(2.5)\\
&= \mathrm{tr}_A\left(M_A(\alpha)(\sum_{ijkl} \delta_{jl}\alpha_{ij}\overline{\alpha_{kl}}|i\rangle_A\langle k|)\right) &(2.6)\\
&= \mathrm{tr}_A\left(M_A(\alpha)\sum_{ijk} \alpha_{ik}\overline{\alpha_{jk}}|i\rangle_A\langle j|\right) &(2.7)\\
&= \mathrm{tr}_A\left(M_A(\alpha)\mathrm{tr}_B(|\Psi\rangle_{AB}\langle\Psi|)\right). &(2.8)
\end{aligned}$$

If we let $\mathrm{tr}_B(|\Psi\rangle_{AB}\langle\Psi|) = \rho_A$,

$$\mathrm{tr}((M_A(\alpha) \otimes I)|\Psi\rangle_{AB}\langle\Psi|) = \mathrm{tr}_A(M_A(\alpha)\rho_A). \tag{2.9}$$

It means that when a measurement of an observable M_A is performed on ρ_A, the probability of each outcome α is exactly same to the original measurement, that is, both of them have the same probability distribution.

Remind that a useful practical way to distinguish and identify a quantum state is to compare the probability distribution by a measurement. Since it is impossible to distinguish the both measurement cases as shown in Eqn (2.9), even though ρ_A is not in the form of Axiom 1, it is reasonable to regard ρ_A as another regular form for a subsystem A of $|\Psi\rangle_{AB}$. Similarly $\rho_B = \mathrm{tr}_A(|\Psi\rangle_{AB}\langle\Psi|)$ can be considered as a quantum state for a subsystem B of $|\Psi\rangle_{AB}$.

Note that the state $\rho_A = \sum_{ijk} \alpha_{ik}\overline{\alpha_{jk}}|i\rangle_A\langle j|$ satisfies the following three properties.

(i) Unit trace : $\mathrm{tr}(\rho_A) = \sum_{ijk} \delta_{ij}\alpha_{ik}\overline{\alpha_{jk}} = \sum_{ij} |\alpha_{ij}|^2 = 1$

(ii) Hermicity : $\rho_A = \rho_A^\dagger$

(iii) Positivity : For any state $|\psi\rangle$, $\langle\psi|\rho_A|\psi\rangle = \sum_k |\sum_i \alpha_{ik}\langle\psi|i\rangle|^2 \geq 0$

When a state satisfies the above three properties, it is called a *density operator*. Note that the positivity of an operator in a complex vector space also implies the hermicity, and thus, from the fact (i), the density operator can always be in the form of

$$\rho = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|,$$

where $\sum_k \lambda_k = 1$ and $|\psi_k\rangle$'s are mutually orthogonal. We call this form the *spectral decomposition* of ρ.

For two-dimensional Hilbert space, the density operators ρ has an explicit form of a linear combination of Pauli operators,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as follows

$$\rho(\overrightarrow{n}) = \frac{1}{2}(I + \overrightarrow{n} \cdot \overrightarrow{\sigma}) \tag{2.10}$$

$$= \frac{1}{2}(I + n_1\sigma_x + n_2\sigma_y + n_3\sigma_3) \tag{2.11}$$

$$= \frac{1}{2}\begin{pmatrix} 1 + n_3 & n_1 - in_2 \\ n_1 + in_2 & 1 - n_3 \end{pmatrix} \tag{2.12}$$

where $\det\rho(\overrightarrow{n}) = \frac{1}{4}(1 - |\overrightarrow{n}|^2)$. If $|\overrightarrow{n}| = 1$, then $\rho(\overrightarrow{n})$ is a one-dimensional projection and it is natural to regard it as a pure state. If $|\overrightarrow{n}| < 1$, then $\rho(\overrightarrow{n})$ has two positive eigenvalues, that is, it is a mixture of two one-dimensional projections onto the corresponding eigenvectors. So, when $|\overrightarrow{n}| < 1$, we call the density operator a *mixed* state.

The mixed state contains a probabilistic distribution in itself and represents an expected state of an ensemble. In fact, the mixed state is generated differently from its ensemble. However, there is no way to distinguish them from each other. For example, suppose that Alice prepares and sends two pure states $|0\rangle$ and $|1\rangle$ with the equal probability to Bob to realize an ensemble of a density operator $I/2$. At a later time, when Bob performs a measurement $M = \{M(\alpha)\}$ to distinguish them, for all outcomes α, both cases have the same probability as follows

$$\frac{1}{2}\mathrm{tr}(M(\alpha)|0\rangle\langle 0|) + \frac{1}{2}\mathrm{tr}(M(\alpha)|1\rangle\langle 1|) \quad = \quad \mathrm{tr}\left(M(\alpha)(\frac{I}{2})\right). \qquad (2.13)$$

Surprisingly, the density operator $I/2$ also implies another ensemble which is prepared by using two states $|+\rangle$ and $|-\rangle$ instead of $|0\rangle$ and $|1\rangle$. Therefore, Bob cannot distinguish two uniformly distributed ensemble $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$.

This fact is one of the most important difference between quantum mechanics and classical mechanics, and it has been broadly applied to quantum cryptographic protocols to guarantee its unconditional security, such as quantum key distribution, quantum secret sharing, quantum bit commitment, quantum oblivious transfer, and so on. More generally, it is also closely related to the Heisenberg's uncertainty principle which tells that it is impossible to estimate two non-commutative observables simultaneously by any kind of measurements.

2.3 Quantum entanglement

For a given bipartite two qubit state $|\Psi\rangle_{AB} = \sum_{ij} \alpha_{ij}|i\rangle_A \otimes |j\rangle_B$, there exist two cases for the ranks r_A and r_B of subsystems ρ_A and ρ_B. Precisely, $r_A = r_B = 1$ and $r_A = r_B = 2$. (*cf.* (r_A, r_B) can be neither $(1, 2)$ nor $(2, 1)$. Of course, in a higher dimensional system, the rank can be greater than 2, and all the following facts hold.)

If $r_A = r_B = 1$, then the bipartite pure state $|\Psi\rangle_{AB}$ can be factorized into the tensor product of two pure states like that $|\Psi\rangle_{AB} = |\psi\rangle_A \otimes |\phi\rangle_B$. This case shows us the very classical property that any quantum operation on each particle makes no effects on the result of a quantum operation on the opposite particle at a distance.

However, when $r_A = r_B = 2$, we face up with an amazing physical phenomenon, so-called *non-locality*, of quantum mechanics. In this case ρ_A and ρ_B each should have the following spectral decompositions

$$\rho_A = \sum_{i=1}^{2} \lambda_i |\psi_i\rangle_A \langle\psi_i|$$

and

$$\rho_B = \sum_{i=1}^{2} \lambda_i |\phi_i\rangle_B \langle\phi_i|.$$

Moreover, $|\Psi\rangle_{AB}$ can be decomposed as

$$|\Psi\rangle_{AB} = \sum_{i=1}^{2} \sqrt{\lambda_i} |\psi_i\rangle_A \otimes |\phi_i\rangle_B$$

which is called the *Schmidt decomposition* and is the quantum analogue of the *singular value decomposition* of a linear transformation.

This kind of state provides a perfect classical correlation between the systems A and B far apart from each other. If Alice obtains $|\psi_i\rangle$ as a result of an orthogonal measurement $\{M_i = |\psi_i\rangle\langle\psi_i|\}$ on subsystem A, then Bob's measurement result should be $|\phi_i\rangle$ when he performs an orthogonal measurement $\{N_i = |\phi_i\rangle\langle\phi_i|\}$ on the opposite subsystem B.

Furthermore, it does not depend on the measurement direction, because the bipartite state $|\Psi\rangle_{AB}$ can be also rewritten by

$$\begin{aligned} |\Psi\rangle_{AB} &= (U_A \otimes V_B)|\Psi\rangle_{AB} \quad (2.14)\\ &= \sum_i \sqrt{\lambda_i}(U_A|\psi_i\rangle_A) \otimes (V_B|\phi_i\rangle_B), \quad (2.15)\end{aligned}$$

where

$$U_A = DV_B^* D^{-1} \quad (2.16)$$

for a diagonal matrix $D = diag(r_0, r_1)$.

This fact can be proved for an arbitrary n-dimensional bipartite state $|\Psi\rangle_{AB}$ as follows. Without loss of generality, let the Schmidt decomposition of $|\Psi\rangle_{AB}$ be in the form of

$$|\Psi\rangle_{AB} = \sum_{i=1}^{n} r_i |i\rangle_A |i\rangle_B$$

and apply two unitary operations $U_A \otimes I$ and $I \otimes V_B$ to $|\Psi\rangle_{AB}$, where U_A and V_B is defined by

$$U_A = \sum_{\alpha,\beta=1}^{n} u_{\alpha\beta}|\alpha\rangle\langle\beta|$$

and

$$V_B = \sum_{\gamma,\delta=1}^{n} v_{\gamma\delta}|\gamma\rangle\langle\delta|.$$

Then the resulting states are like that

$$(U_A \otimes I)|\Psi\rangle_{AB} = \sum_{\alpha,\beta} r_\beta u_{\alpha\beta}|\alpha\rangle_A|\beta\rangle_B$$

and

$$(I \otimes V_B)|\Psi\rangle_{AB} = \sum_{\gamma,\delta} r_\delta u_{\gamma\delta} |\delta\rangle_A |\gamma\rangle_B.$$

Therefore, if $(U_A \otimes I)|\Psi\rangle_{AB} = (I \otimes V_B)|\Psi\rangle_{AB}$, the following equation should be satisfied

$$r_\beta u_{\alpha\beta} = r_\alpha u_{\beta\alpha} \tag{2.17}$$

for all α and β. The left-hand side and right-hand side of Eqn. (2.17) can be reexpressed by

$$(U_A R)_{\alpha\beta} = (RV_B)_{\beta\alpha} = (V_B^T R)_{\alpha\beta}$$

and thus $U_A R = RV_B^T$. If all r_i's are positive, that is, R is invertible, then

$$U_A = RV_B^T R^{-1}.$$

In result,

$$\begin{aligned} (I \otimes I)|\Psi\rangle_{AB} &= \left((RV_B^T R^{-1})^{-1} \otimes V_B\right)|\Psi\rangle_{AB} && (2.18)\\ &= (RV_B^* R^{-1} \otimes V_B)|\Psi\rangle_{AB}. && (2.19)\end{aligned}$$

In particular, if the density operators ρ_A and ρ_B have the degeneracy, that is, $\lambda_i = \lambda_j$ for some $i \neq j$, then both of U_A and V_B can be unitary operators and thus we can make another correlation with the rotated orthogonal measurements $U_A M_i$ and $V_B N_i$. Of course, when the density operators are non-degenerate, we should implement the generalized measurement such as a positive operator valued measurement (POVM) to make a similar effect. When the rank is more than 1 as described above, the bipartite pure state is called an *entangled* state. The quantum entanglement is also a surprising quantum phenomenon which cannot be found in classical mechanics.

2.4 The von Neumann entropy

In this section, we introduce the von Neumann entropy as quantum analogue of the Shannon entropy and its properties.

Definition 2.4.1 (The von Neumann entropy)**.** *The entropy $S(\rho)$ of the state ρ with the spectral decomposition $\rho = \sum_x p_X(x)|x\rangle\langle x|$*

$$\begin{aligned} S(\rho) &= -Tr\{\rho \log \rho\} && (2.20) \\ &= -\sum_x p_X(x) \log p_X(x) && (2.21) \end{aligned}$$

The *The von Nenmann entropy* has a special relation to the eigenvalues of the density operator as the entropy $H(X)$ of a random variable X with probability distribution $p_X(x)$.

We now exhibit several mathematical properties of the quantum entropy: Nonnegativity, Minimum value, Maximum value, Concavity, Unitary Invariance.

Theorem 2.4.1 (Nonnegativity)**.** *The von Neumann entropy $S(\rho)$ is nonnegative for any density operator ρ :*

$$S(\rho) \geq 0 \tag{2.22}$$

Theorem 2.4.2 (Minimum value)**.** *The minimum value of the von Neumann entropy is zero, and it occurs when the density operator is a pure state.*

Theorem 2.4.3 (Maximum value)**.** *The minimum value of the von Neumann entropy is logD where D is the dimension of the system, and it occurs for the maximally mixed state.*

Theorem 2.4.4 (Concavity)**.** *The von Neumann entropy $S(\rho)$ is concave in a density operators :*

$$S(\rho) \geq \sum_x p_X(x) S(\rho_x) \tag{2.23}$$

where $\rho = \sum_x p_X(x)\rho_x$.

Theorem 2.4.5 (Unitary invariance)**.** *The von Neumann entropy $S(\rho)$ of density operator is invariant under unitary operations on it :*

$$S(\rho) = S(U\rho U^\dagger). \tag{2.24}$$

Definition 2.4.2 (The joint von Neumann entropy)**.** *The entropy $S(\rho^{AB})$ of the density operator ρ^{AB} for a bipartite system AB follows naturally from the definition of von Neumann entropy:*

$$S(\rho^{AB}) = -Tr\{\rho^{AB} \log \rho^{AB}\} \tag{2.25}$$

Theorem 2.4.6 (The joint entropy of classical-quantum state)**.** *The joint entropy of classical-quantum state $\rho^{XB} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \rho_x^B$ is as follows:*

$$S(XB) = H(X) + \sum_x p_X(x) S(\rho_x^B) \tag{2.26}$$

where $H(X)$ is the entropy of a random variable with distribution $p_X(x)$

The mutual information is the standard measure of correlation in the classical system. And such a quantity plays a useful role in measuring classical and quantum correlation in the quantum system as well.

Definition 2.4.3 (Quantum mutual information)**.** *The Quantum mutual information of a bipartite quantum state ρ^{AB} is as follows:*

$$\mathcal{I}(A;B) = S(A) + S(B) - S(AB). \tag{2.27}$$

Theorem 2.4.7 (Nonnegativity of Quantum mutual information)**.** *The Quantum mutual information $\mathcal{I}(A;B)$ of any bipartite quantum state ρ^{AB} is nonnegative:*

$$\mathcal{I}(A;B) \geq 0 \tag{2.28}$$

Definition 2.4.4 (Quantum relative entropy)**.** *The Quantum relative entropy $D(\rho||\sigma)$ between two states ρ and σ is as follows:*

$$D(\rho||\sigma) = Tr\{\rho(\log\rho - \log\sigma)\}. \tag{2.29}$$

Similar to the classical case, we can intuitively think of it as a distance measure between quantum states. But in the mathematical sense it is not strictly a distance measure. Because it is not symmetric and does not hold a triangle inequality. Nevertheless, the quantum relative entropy is always nonnegative.

Theorem 2.4.8 (Nonnegativity of Quantum Relative Entropy)**.** *The Quantum relative entropy $D(\rho||\sigma)$ is nonnegative for any two density operators ρ and σ :*

$$D(\rho||\sigma) \geq 0 \tag{2.30}$$

2.5 The Tsallis entropy

Definition 2.5.1 (The Tsallis entropy)**.** *Using generalized logarithmic function, called q-logarithm,*

$$\ln_q x \;=\; \frac{x^{1-q}-1}{1-q}. \tag{2.31}$$

one can define Tsallis entropy *of rank q as*

$$\begin{aligned} T_q(P) &= -\sum_j p_j^q \ln_q p_j && (2.32)\\ &= \frac{1}{q-1}(1-\sum_j p_j^q). && (2.33)\end{aligned}$$

Note that $q \to 1$ corresponds to Shannon entropy.

Definition 2.5.2 (Tsallis conditional entropy and joint entropy)**.** *For the conditional probability $p(x|y) \equiv p(X=x|Y=y)$ and the joint probability $p(x,y) \equiv p(X=x, Y=y)$, we define Tsallis conditional entropy and joint entropy by*

$$T_q(X|Y) \equiv -\sum_{x,y} p(x,y)^q \ln_q p(x|y), (q \neq 1) \tag{2.34}$$

and

$$T_q(XY)(=T_q(X,Y)) \equiv -\sum_{x,y} p(x,y)^q \ln_q p(x,y), (q \neq 1). \tag{2.35}$$

Propostion 2.5.1.

$$T_q(XY) = T_q(X) + T_q(Y|X) \tag{2.36}$$

(Therefore immediately $T_q(X) \leq T_q(XY)$.)

Lemma 2.5.1 (Chain rule)**.**

$$(1)T_q(X,Y,Z) = T_q(X,Y|Z) + T_q(Z) \tag{2.37}$$

$$(2)T_q(X,Y|Z) = T_q(X|Z) + T_q(Y|X,Z) \tag{2.38}$$

Definition 2.5.3 (Tsallis relative entropy)**.** *For two probability distributions $u(x)$ and $v(x)$, and $q \geq 0$. we define Tsallis relative entropy by*

$$D_q(X \parallel Y) \equiv -\sum_x u(x) \ln_q \frac{v(x)}{u(x)} = \sum_x \frac{u(x)^q v(x)^{1-q} - 1}{1-q} \tag{2.39}$$

Definition 2.5.4 (Tsallis mutual information)**.** *For two random variables X and Y, we define the Tsallis mutual information as the difference between Tsallis entropy and Tsallis conditional entropy such that*

$$I_q(X;Y) \equiv T_q(X) - T_q(X|Y) = T_q(X) + T_q(Y) - T_q(XY) \tag{2.40}$$

Definition 2.5.5 (Quantum Tsallis entropy)**.** *For density operators ρ and $q > 0,\ q \neq 1$, we define the quantum Tsallis entropy as*

$$S_q(\rho) \equiv -Tr\rho^q \ln_q \rho = \frac{1 - Tr(\rho^q)}{q-1} \tag{2.41}$$

In the framework of composite quantum systems, The quantum Tsallis entropy can be defined in perfect analogue to the classic case as
$S_q(\rho^{AB}) \equiv -Tr(\rho^{AB})^q \ln_q \rho^{AB} = \frac{1-Tr((\rho^{AB})^q)}{q-1}$, $q > 0,\ q \neq 1$.
The quantum Tsallis entropy function $S_q(x)$ is nonnegative, concave and, if ρ^{AB} is pure then $S_q(A) = S_q(B)$.

Chapter 3

Quantum discord

In this Chapter, we summarize Quantum discord (QD) proposed by H. Ollivier and W. H. Zurek (2001) and Global quantum discord (GQD) proposed by Rulli and Sarandy (2011) [14, 18]. And we enumerate its related results.

3.1 Quantum discord

3.1.1 Quantum discord in bipartite system

In classic system, the mutual information has two equivalent expressions. But not in quantum system. This difference defines the quantum discord.

The first, in classic system, the correlation between two random variables X and Y is measured by the mutual information :

$$J(X:Y) = H(X) - H(X|Y) \tag{3.1}$$

where $H(X|Y) = \sum_y H(X|Y = y)$ is conditional entropy of given X and Y. Using the Bayes rule, one can show that $H(X|Y) = -H(Y) + H(X,Y)$. So we can obtain another classically identical expression for the mutual information :

$$I(X:Y) = H(X) + H(Y) - H(X,Y) \tag{3.2}$$

We know that there is no difference between $I(X:Y)$ and $J(X:Y)$. i.e. $I(X:Y) = J(X:Y)$.

All the ingredients involved in the definition of $\mathcal{I}$ are easily generalized to deal with arbitrary quantum system by replacing the classical probability distributions by the appropriate density matrices ρ^A, ρ^B and ρ^{AB} and the Shannon entropy by the von Neumann entropy.:

$$\mathcal{I}(A:B) = S(A) + S(B) - S(A,B) \tag{3.3}$$

But the generalization of another expression $\mathcal{J}$ in quantum systems is not easy, because the generalization of the conditional entropy $S(X|Y)$ in quantum systems requires us to specify the state of A given the state of B. Such statement in quantum theory is ambiguous until the to-be-measured set of states A is selected. The state of A, after the outcome corresponding to B_j has been detected, is $\rho^{A|\Pi_j^B} = \Pi_j^B \rho^{A,B} \Pi_j^B / tr \Pi_j^B \rho^{A,B} \Pi_j^B$, with probability $p_j = tr \Pi_j^B \rho^{A,B} \Pi_j^B$, $\{\Pi_j^B\}$ is the set of one dimensional projectors. $S(\rho^{A|\Pi_j^B})$ is the missing information about A. The entropies $S(\rho^{A|\Pi_j^B})$, weighted by probabilities, p_j, yield to the conditional entropy of A given the complete measurement $\{\Pi_j^B\}$ on B.

$$S(A|\{\Pi_j^B\}) = \sum_j p_j S(\rho^{A|\Pi_j^B}) \tag{3.4}$$

This leads to the following quantum generalization of $\mathcal{J}$:

$$\mathcal{J}(A:B)_{\{\Pi_j^B\}} = S(A) - S(A|\{\Pi_j^B\}) \tag{3.5}$$

By applying changes, they has defined a quantum discord as follows.

Definition 3.1.1 (Quantum Discord in bipartite system)**.** *The quantum discord is the minimalization of the difference between $\mathcal{I}$ and $\mathcal{J}$:*

$$QD^B(\rho^{AB}) = QD^B(A:B) \tag{3.6}$$

$$= \mathcal{I}(\rho^{AB}) - \sup_{\{\Pi_i^B\}} \mathcal{J}_{\{\Pi_i^B\}}(\rho^{AB}) \tag{3.7}$$

And then they have shown a nonnegativity of the quantum discord.

Theorem 3.1.1 (Nonnegativity of Quantum Discord)**.** *The quantum discord $QD^A(\rho^{AB})$ and $QD^B(\rho^{AB})$ is nonnegative for any measurement $\{\Pi_i^A\}$, $\{\Pi_i^B\}$.*

$$QD^A(\rho^{AB}) \geq 0 \quad \textit{and} \quad QD^B(\rho^{AB}) \geq 0 \tag{3.8}$$

S. Luo and S. Fu obtained an alternative expression of the quantum discord $QD^B(\rho^{AB})$ as the minimal loss of correlations caused by the non-selective von Neumann projective measurement performed on the system B [15].

$$QD^B(\rho^{AB}) = \inf_{\{\Pi_i^B\}} [\mathcal{I}(\rho^{AB}) - \mathcal{I}(\Phi_{\{\Pi_i^B\}}(\rho^{AB}))] \tag{3.9}$$

where $\Phi_{\{\Pi_i^B\}}(\rho^{AB}) = \sum_i (I \otimes \Pi_i^B)\rho^{AB}(I \otimes \Pi_i^B)$.

3.1.2 Quantum discord in multipartite system

In this section, we summarize Okrasa and Walczak's results about the quantum discord in multipartite system [20]. Let us consider N quantum systems, $A_1, \cdots, A_N$, in a state $\rho^{\mathbf{A}}$. The quantum mutual information of a state $\rho^{\mathbf{A}}$ is given by

$$\mathcal{I}(\rho^{\mathbf{A}}) = \sum_i S(\rho^{A_i}) - S(\rho^{\mathbf{A}}) \tag{3.10}$$

The quantum conditional entropy, $S(\rho^{[A_k]|A_k}) = S(\rho^{\mathbf{A}}) - S(\rho^{A_k})$, allows one to rewrite the quantum mutual information in the following form

$$\mathcal{I}(\rho^{\mathbf{A}}) = \sum_{i \neq k} S(\rho^{A_i}) - S(\rho^{[A_k]|A_k}) \tag{3.11}$$

where $[A_k]$ stands for $A_1 \cdots A_{k-1} A_{k+1} \cdots A_N$.

The von Neumann measurement $\{\Pi_i^{A_k}\}$, corresponding to outcomes i, is performed then the post-measurement joint state of the systems $[A_k]$ is given by

$$\rho^{[A_k]|i} = Tr_{A_k}[\Pi_i^{A_k} \rho^{\mathbf{A}} \Pi_i^{A_k}]/p_i^{A_k} \tag{3.12}$$

where $P_i^{A_k} = (I \otimes \cdots \otimes \Pi_i^{A_k} \otimes \cdots \otimes I)$ and $p_i^{A_k} = Tr[P_i^{A_k} \rho^{\mathbf{A}}]$. The von Neumann entropies $S(\rho^{[A_k]|i})$, weighted by probabilities $p_i^{A_k}$, lead to the quantum conditional entropy of the systems $[A_k]$ given the complete measurement $\{\Pi_i^{A_k}\}$ on the system A_k

$$S_{\{\Pi_i^{A_k}\}}(\rho^{[A_k]|i}) = \sum_i p_i^{A_k} S(\rho^{[A_k]|i}), \tag{3.13}$$

and thereby the quantum mutual information, induced by the von Neumann measurement performed on the system A_k, is defined by

$$\mathcal{J}_{\{\Pi_i^{A_k}\}}(\rho^{\mathbf{A}}) = \sum_{i \neq k} S(\rho^{A_i}) - S_{\{\Pi_i^{A_k}\}}(\rho^{[A_k]|A_k}). \tag{3.14}$$

Definition 3.1.2 (Quantum Discord in multipartite system)**.** *The quantum discord is the minimalization of the difference between I and J :*

$$
\begin{aligned}
QD^{A_k}(\rho^{\mathbf{A}}) &= \mathcal{I}(\rho^{\mathbf{A}}) - \sup_{\{\Pi_i^{A_k}\}} \mathcal{J}_{\{\Pi_i^{A_k}\}}(\rho^{\mathbf{A}}) && (3.15)\\
&= \inf_{\{\Pi_i^{A_k}\}} [\mathcal{I}(\rho^{\mathbf{A}}) - \mathcal{I}(\Phi_{\{\Pi_i^{A_k}\}}(\rho^{\mathbf{A}}))] && (3.16)\\
&= S(\rho^{A_k}) - S(\rho^{\mathbf{A}}) + \inf_{\{\Pi_i^{A_k}\}} \sum_i p_i^{A_k} S(\rho^{[A_k]|i}). && (3.17)
\end{aligned}
$$

where $\Phi_{\{\Pi_i^{A_k}\}}(\rho^{\mathbf{A}}) = \sum_i P_i^{A_k} \rho^{\mathbf{A}} P_i^{A_k}$.

3.2 Global Quantum discord

In this section, we give full detail of global quantum discord(GQD) proposed by Rulli and Sarandy [18] and an equivalent expression for GQD, proposed by Jianwei Xu [22] in multipartite states.

The mutual information $\mathcal{I}(\rho^{AB})$ can express as the relative entropy between ρ^{AB} and $\rho^{A} \otimes \rho^{B}$, i.e.

$$\mathcal{I}(\rho^{AB}) = D(\rho^{AB}||\rho^{A} \otimes \rho^{B}). \tag{3.18}$$

In order to express the measurement-induced quantum mutual information $\mathcal{J}(\rho^{AB})$ in terms of relative entropy, we consider a non-selective von-Neumann measurement on part B of ρ^{AB}, which yields

$$\Phi^{B}(\rho^{AB}) = \sum_{j}(I \otimes \Pi_{j}^{B})\rho^{AB}(I \otimes \Pi_{j}^{B}) = \sum_{j} p_{j}\rho^{A|j} \otimes |b_{j}\rangle\langle b_{j}|. \tag{3.19}$$

Moreover, tracing over the variables of the subsystem A, we obtain

$$\Phi^{B}(\rho^{B}) = Tr_{A}(\Phi^{B}(\rho^{AB})) = \sum_{j} p_{j}|b_{j}\rangle\langle b_{j}|, \tag{3.20}$$

where we have used that $Tr_{A}(\rho^{A|j}) = 1$. Then, by expressing the entropies $S(\Phi^{B}(\rho^{AB}))$ and $S(\Phi^{B}(\rho^{B}))$ as

$$S(\Phi^{B}(\rho^{AB})) = H(p) + \sum_{j} p_{j}S(\rho^{A|j}) \tag{3.21}$$

and

$$S(\Phi^{B}(\rho^{B})) = H(p), \tag{3.22}$$

with $H(p)$ denoting the Shannon entropy.
We can rewrite $\mathcal{J}^{B}(\rho^{AB})$ as

$$\mathcal{J}^{B}(\rho^{AB}) = S(\rho^{A}) - \sum_{j} p_{j}S(\rho^{A|j}) = S(\rho^{A}) + S(\Phi^{B}(\rho^{B})) - S(\Phi^{B}(\rho^{AB})). \tag{3.23}$$

Finally, we can obtain an equality follow as:

$$\mathcal{J}^B(\rho^{AB}) = \mathcal{I}^B(\Phi^B(\rho^{AB})). \tag{3.24}$$

In Def 3.2.1, we introduce the global quantum discord and its equivalent expression obtained by applying equation of 3.24.

Definition 3.2.1 (Global Quantum Discord in bipartite system)**.** *The global quantum discord $GQD(\rho^{AB})$ for an arbitrary bipartite state ρ^{AB} under a set of local measurements $\{\Pi_i^A \otimes \Pi_i^B\}$ is defined as*

$$\begin{aligned} GQD(\rho^{AB}) &= \mathcal{I}(\rho^{AB}) - \sup_{\{\Pi_i^A\otimes\Pi_i^B\}} \mathcal{J}_{\{\Pi_i^A\otimes\Pi_i^B\}}(\rho^{AB}) & (3.25)\\ &= \inf_{\{\Pi_i^A\otimes\Pi_i^B\}} [\mathcal{I}(\rho^{AB}) - \mathcal{I}(\Phi_{\{\Pi_i^A\otimes\Pi_i^B\}}(\rho^{AB}))]. & (3.26) \end{aligned}$$

We easily obtain a generalization of global quantum discord to multipartite states.

Definition 3.2.2 (Global Quantum Discord in multipartite system)**.** *The global quantum discord $GQD(\rho^{A_1\cdots A_N})$ for an arbitrary bipartite state $\rho^{A_1\cdots A_N}(= \rho^{\mathbf{A}})$ under a set of local measurements $\{\Pi_{i_1}^{A_1} \otimes \cdots \otimes \cdots \otimes \Pi_{i_N}^{A_N}\}$ is defined as*

$$\begin{aligned} GQD(\rho^{A_1\cdots A_N}) &= \mathcal{I}(\rho^{\mathbf{A}}) - \sup_{\{\Pi_k\}} \mathcal{J}_{\{\Pi_k\}}(\rho^{\mathbf{A}}) & (3.27)\\ &= \inf_{\{\Pi_k\}} [\mathcal{I}(\rho^{\mathbf{A}}) - \mathcal{I}(\Phi_{\{\Pi_k\}}(\rho^{\mathbf{A}}))]. & (3.28) \end{aligned}$$

where $\Phi_{\{\Pi_k\}}(\rho^{\mathbf{A}}) = \sum_k \Pi_k \rho^{\mathbf{A}} \Pi_k$ with $\Pi_k = \Pi_{j_1}^{A_1} \otimes \cdots \otimes \cdots \otimes \Pi_{j_N}^{A_N}$ and k denoting the index string $(j_1, \cdots, j_N)$.

Theorem 3.2.1 (Nonnegativity of GQD). *The global quantum discord* $GQD(\rho^{\mathbf{A}})$ *is nonnegative, i.e.,* $GQD(\rho^{\mathbf{A}}) \geq 0$.

For a special case, we consider a state $\rho^{\mathbf{A}}$ whose reduced states $\rho^{A_1}, \cdots, \rho^{A_N}$ are all proportional to identity operator.

Theorem 3.2.2 (GQD for special case). *An* N*-partistate* $\rho^{\mathbf{A}}$ *whose reduced states* $\rho^{A_1}, \cdots, \rho^{A_N}$ *are all proportional to identity operator, then the GQD of* $\rho^{\mathbf{A}}$ *can be expressed as*

$$GQD(\rho^{\mathbf{A}}) = -S(\rho^{\mathbf{A}}) + \inf_{\{\Pi_k\}} S(\Phi_{\Pi_k}(\rho^{\mathbf{A}})). \tag{3.29}$$

Chapter 4

Quantum discord derived from Tsallis entropy

4.1 Quantum discord derived from Tsallis entropy

In this section, we display the generalized quantum discord so as to encompass Tsallis entropy defined by Majtey et al., and Jurkowski [23, 24].

The first, we introduce two kinds of expression of mutual information derived from Tsallis entropy, classically equivalent, by using the relation between a conditional entropy and a joint entropy:

$$I_q(X;Y) = T_q(X) + T_q(Y) - T_q(X,Y), \tag{4.1}$$

and

$$J_q(X;Y) = T_q(X) - T_q(X|Y) \tag{4.2}$$

We can obtain easily generalized defining appropriate density matrices for the quantum systems, ρ^A, ρ^B and ρ^{AB}, and applying then the Tsallis quantum entropy $S_q(\rho)$. The $\mathcal{I}_q(X;Y)$, $\mathcal{J}_q^B(X;Y)$ in quantum system can be defined as:

$$\mathcal{I}_q(A;B) = \mathcal{I}_q(\rho^{AB}) = S_q(\rho^A) + S_q(\rho^B) - S_q(\rho^{AB}) \tag{4.3}$$

and

$$\mathcal{J}_q(A;B)_{\{\Pi_i^B\}} = \mathcal{J}_q^B(\rho^{AB}) = S_q(\rho^A) - S_q(\rho^A|\{\Pi_j^B\}), \tag{4.4}$$

where $S_q(\rho^A|\{\Pi_j^B\}) = \sum_j p_j^q S_q(\rho^{A|\{\Pi_j^B\}})$, with the state of A given, once measurement is performed by

$$\rho^{A|\{\Pi_j^B\}} = \Pi_j^B \rho^{AB} \Pi_j^B / Tr_{AB} \Pi_j^B \rho^{AB} \quad \textit{and} \quad p_j = Tr_{AB} \Pi_j^B \rho^{AB}. \tag{4.5}$$

Definition 4.1.1 (q-Quantum Discord)**.** *The q-quantum discord $GQd_q^B(\rho^{AB})$ for an arbitrary bipartite state ρ^{AB} under a set of local projective measurements $\{\Pi_j^B\}$ is defined as*

$$GQd_q^B(\rho^{AB}) = \mathcal{I}_q(\rho^{AB}) - \sup_{\{\Pi_i^B\}} \mathcal{J}_q^B(\rho^{AB}) \tag{4.6}$$

We normalize this measure via a trivial re-scaling in order to compare, in an adequate way, different quantities:

$$GQD_q^B(\rho^{AB}) = \frac{q-1}{1-2^{1-q}} GQd_q^B(\rho^{AB}). \tag{4.7}$$

Majtey et al. has shown the following thm by a numerical verification[1] of the concavity of the conditional entropy for $q \in (0,1)$ [23].

Theorem 4.1.1 (Nonnegativity of q-Quantum Discord)**.** *The q-Quantum Discord $GQD_q^B(\rho^{AB})$ has nonnegativity for $q \in (0,1)$.*

And they have shown that an order-relation for quantum states based on discord lacks unicity because it definitely depends on the quantifier one chooses to employ. This means that q-QD functionals corresponding to different values of q measure different aspects of quantumness of correlation [23].

[1] They checked the nonnegativity of q-GQD for 10^6 random states by simulating

4.2 Global quantum discord derived from Tsallis entropy

In this section, we publish our results.

4.2.1 Global quantum discord derived from Tsallis entropy

In order to express the measurement-induced quantum mutual information $J(\rho^{AB})$, we consider a non-selective von-Neumann measurement on part B of ρ^{AB}, which yields

$$\Phi^B(\rho^{AB}) = \sum_j (I \otimes \Pi_j^B)\rho^{AB}(I \otimes \Pi_j^B) = \sum_j p_j \rho^{A|j} \otimes |b_j\rangle\langle b_j|. \tag{4.8}$$

Moreover, tracing over the variables of the subsystem A, we obtain

$$\Phi^B(\rho^B) = Tr_A(\Phi^B(\rho^{AB})) = \sum_j p_j |b_j\rangle\langle b_j|, \tag{4.9}$$

where we have used that $Tr_A(\rho^{A|j}) = 1$. Then, by expressing the entropies $S_q(\Phi^B(\rho^{AB}))$ and $S_q(\Phi^B(\rho^B))$ as

$$\begin{aligned}
S_q(\Phi^B(\rho^{AB})) &= \frac{1 - tr((\sum_j p_j \rho^{A|j} \otimes |b_j\rangle\langle b_j|))^q)}{q-1} && (4.10)\\
&= \frac{1 - tr(\sum_j p_j^q (\rho^{A|j})^q}{q-1}) && (4.11)\\
&= \frac{1 - \sum_j p_j^q + \sum_j p_j^q - tr(\sum_j p_j^q (\rho^{A|j})^q)}{q-1} && (4.12)\\
&= \frac{1 - \sum_j p_j^q}{q-1} + \frac{\sum_j p_j^q - tr(\sum_j p_j^q (\rho^{A|j})^q)}{q-1} && (4.13)\\
&= \frac{1 - \sum_j p_j^q}{q-1} + \sum_j p_j^q \frac{1 - tr((\rho^{A|j})^q)}{q-1} && (4.14)\\
&= S_q(\Phi^B(\rho^B)) + \sum_j p_j^q S_q(\rho^{A|j}). && (4.15)
\end{aligned}$$

Therefore, we can obtain the relation between $\mathcal{J}_q^B(\rho^{AB})$ and $\mathcal{I}_q(\Phi^B(\rho^{AB}))$.

$$\begin{aligned}
\mathcal{J}_q^B(\rho^{AB}) &= S_q(\rho^A) - \sum_j p_j^q S_q(\rho^{A|j}) \\
&= S_q(\rho^A) + S_q(\Phi^B(\rho^B)) - S_q(\Phi^B(\rho^{AB})) \\
&= \mathcal{I}_q(\Phi^B(\rho^{AB})). \qquad (4.16)
\end{aligned}$$

Therefore, by using this fact, we provide an equivalent expression for q-quantum discord.

Definition 4.2.1 (q-Quantum Discord)**.** *The q-quantum discord $QD_q^B(\rho^{AB})$ for an arbitrary bipartite state ρ^{AB} under a set of local projective measurements $\{\Phi^B\}$ is defined as*

$$QD_q^B(\rho^{AB}) = \frac{q-1}{1-2^{1-q}}[\mathcal{I}_q(\rho^{AB}) - \sup_{\{\Phi^B\}} \mathcal{I}_q^B(\Phi^B(\rho^{AB}))] \qquad (4.17)$$

We propose a global measure, called q-Global Quantum Discord, for quantum correlation in multipartite systems, which is obtained by suitably substituting global measurement for local measurement in mutual information.

Definition 4.2.2 (q-Global Quantum Discord)**.** *The q-global quantum discord $GQD_q(\rho^{A_1\cdots A_N})(= QD_q(\rho^{A_1\cdots A_N}))$ for an arbitrary bipartite state $\rho^{A_1\cdots A_N}(= \rho^{\mathbf{A}})$ under a set of local measurements $\{\Pi_{i_1}^{A_1} \otimes \cdots \otimes \Pi_{i_N}^{A_N}\}(= \{\Phi\}$)is defined as*

$$\begin{aligned}
GQD_q(\rho^{A_1\cdots A_N}) &= \mathcal{I}_q(\rho^{\mathbf{A}}) - \sup_{\{\Phi\}} \mathcal{I}_q(\Phi(\rho^{\mathbf{A}})) \\
&= \sum_{i=1}^{N} S_q(\rho^{A_i}) - S_q(\rho^{\mathbf{A}}) - \sup_{\{\Phi\}}[\sum_{i=1}^{N} S_q(\Phi_i(\rho^{A_i})) - S_q(\Phi(\rho^{\mathbf{A}}))]
\end{aligned}$$

where $\Phi_i = I^{A_1} \otimes \cdots I^{A_{i-1}} \otimes \Pi_{j_i}^{A_i} \otimes I^{A_{i+1}} \cdots \otimes I^{A_N}$.

4.2.2 Properties of q-GQD

In this subsection, we state mathematical facts [26, 27] and our results which is properties, which recovered the results in [18, 19, 22, 23], about q-GQD .

By the Thm 4.1.1, we can numerically show a following Theorem .

Theorem 4.2.1 (Nonnegativity of q-GQD)**.**

$$GQD_q(\rho^{AB}) \;\; \geq \;\; 0 \quad in \quad 0 \leq q < 1 \tag{4.18}$$

$$GQD_q(\rho^{\mathbf{A}}) \;\; \geq \;\; 0 \quad in \quad 0 \leq q < 1 \tag{4.19}$$

Proof. By thm 4.1.1, $QD_q^B(\rho^{AB}) \geq 0$, $QD_q^A(\Phi^B(\rho^{AB})) \geq 0$.

$$\begin{aligned} GQD_q(\rho^{AB}) &= \frac{q-1}{1-2^{1-q}} \inf_{\{\Phi,\Phi^B\}} [\mathcal{I}_q(\rho^{AB}) - \mathcal{I}_q(\Phi^B(\rho^{AB})) \\ &\quad\quad + \mathcal{I}_q(\Phi^B(\rho^{AB})) - \mathcal{I}_q(\Phi(\rho^{AB}))] \\ &= \frac{q-1}{1-2^{1-q}} \inf_{\{\Phi^B\}} [\mathcal{I}_q(\rho^{AB}) - \mathcal{I}_q(\Phi^B(\rho^{AB}))] \\ &\quad\quad + \frac{q-1}{1-2^{1-q}} \inf_{\{\Phi,\Phi^B\}} [\mathcal{I}_q(\Phi^B(\rho^{AB})) - \mathcal{I}_q(\Phi(\rho^{AB}))] \\ &= QD_q^B(\rho^{AB}) + QD_q^A(\Phi^B(\rho^{AB})) \\ &\geq 0. \end{aligned} \tag{4.20}$$

Similarly, $QD_q^{A_k}(\rho^{\mathbf{A}}) \geq 0$, $QD_q^{A_k^C}(\Phi^{A_k}(\rho^{\mathbf{A}})) \geq 0$

$$\begin{aligned} GQD_q(\rho^{\mathbf{A}}) &= \frac{q-1}{1-2^{1-q}} \inf_{\{\Phi,\Phi^{A_k}\}} [\mathcal{I}_q(\rho^{\mathbf{A}}) - \mathcal{I}_q(\Phi^{A_k}(\rho^{\mathbf{A}})) \\ &\quad\quad + \mathcal{I}_q(\Phi^{A_k}(\rho^{\mathbf{A}})) - \mathcal{I}_q(\Phi(\rho^{\mathbf{A}}))] \\ &= \frac{q-1}{1-2^{1-q}} \inf_{\{\Phi^{A_k}\}} [\mathcal{I}_q(\rho^{\mathbf{A}}) - \mathcal{I}_q(\Phi^{A_k}(\rho^{\mathbf{A}}))] \\ &\quad\quad + \frac{q-1}{1-2^{1-q}} \inf_{\{\Phi,\Phi^{A_k}\}} [\mathcal{I}_q(\Phi^{A_k}(\rho^{\mathbf{A}})) - \mathcal{I}_q(\Phi(\rho^{\mathbf{A}}))] \\ &= QD_q^{A_k}(\rho^{\mathbf{A}}) + QD_q^{A_k^C}(\Phi^{A_k}(\rho^{\mathbf{A}})) \\ &\geq 0. \end{aligned} \tag{4.21}$$

where, A_k^C means the other parties of A_k in $\mathbf{A}$ $\square$

Let me consider the special class of N-qubit states whose reduced states ρ^{A_1}, $\cdots$, ρ^{A_N} are all proportional to identity operator. An N-partite state $\rho^{\mathbf{A}}$ whose reduced states ρ^{A_1}, $\cdots$, ρ^{A_N} are all proportional to identity operator, then the q-GQD of $\rho^{\mathbf{A}}$ can be expressed $GQD_q(\rho^{\mathbf{A}}) = -S_q(\rho^{\mathbf{A}}) + \inf_{\Phi} S_q(\Phi(\rho^{\mathbf{A}}))$. By theorem 2.5 and corollary 2.6 in [26], we can show that the nonnegativity of $GQD_q(\rho^{\mathbf{A}})$ hold in $0 \leq q < 1$ for the special class.

We then recall two facts to derive a formula of GQD_q for two classes in the special class. The first if ϕ is symmetric and convex(concave), then ϕ is Schur-convex(concave). The second Tsallis entropy $T_q(X)$ is symmetric and concave. So we obtain easily following lemma by thm of Chap 3 in [27].

Lemma 4.2.1 (Monotonicity of q-entropy function under majorization)**.** *For given* $\{p_1, p_2, \cdots, p_n\}$, $\{q_1, q_2, \cdots, q_n\}$, *satisfy* $1 \geq p_1 \geq \cdots \geq p_n \geq 0$, $\sum_{i=1}^n p_i = 1$, $1 \geq q_1 \geq \cdots \geq q_n \geq 0$, $\sum_{i=1}^n q_i = 1$.

$$\sum_{i=1}^k p_i \leq \sum_{i=1}^k q_i, \ k = 1, \cdots, n \quad \Rightarrow \quad T_q(P) \geq T_q(Q) \tag{4.22}$$

So we can obtain following Theorems which is analogy of Jianwei's process [22].

Theorem 4.2.2 (q-GQD for the N-qubit Werner-GHZ states)**.** *The N-qubit Werner-GHZ states* $\rho = (1-\mu)\frac{I^{\otimes N}}{2^N} + \mu|\psi\rangle\langle\psi|$ *where* I *is* 2×2 *identity operator,* $\mu \in [0,1]$, $|\psi\rangle = (|00\cdots 0\rangle + |11\cdots 1\rangle)/\sqrt{2}$.

$$\begin{aligned} GQD_q(\rho) &= (\frac{1-\mu}{2^N} + \mu)^q \ln_q(\frac{1-\mu}{2^N} + \mu) + (\frac{1-\mu}{2^N})^q \ln_q(\frac{1-\mu}{2^N}) \\ &\quad -2(\frac{1-\mu}{2^N} + \frac{\mu}{2})^q \ln_q(\frac{1-\mu}{2^N} + \frac{\mu}{2}) \end{aligned} \tag{4.23}$$

Theorem 4.2.3 (q-GQD for the N-qubit states)**.** *The N-qubit states $\rho = \frac{1}{2^N}(I^{\otimes N}+c_1\sigma_x^{\otimes N}+c_2\sigma_y^{\otimes N}+c_3\sigma_z^{\otimes N})$ where I is 2×2 identity operator, $\{c_1,\ c_2,\ c_3\}$ are real numbers constrained by $d \in [0,1]$ (N is odd) or $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4 \in [0,1]$ (N is even).*
N is odd

$$GQD_q(\rho) = -\frac{2^{N-1}}{q-1}\{(\frac{1+c}{2^N})^q + (\frac{1-c}{2^N})^q - (\frac{1+d}{2^N})^q - (\frac{1-d}{2^N})^q\} \quad (4.24)$$

where $c = \max\{|c_1|,|c_2|,|c_3|\}$, $d = \sqrt{c_1^2+c_2^2+c_3^2}$.
N is even

$$GQD_q(\rho) = -\frac{2^{N-2}}{q-1}\{2(\frac{1+c}{2^N})^q + 2(\frac{1-c}{2^N})^q - \sum_{j=1}^{4}(\frac{\lambda_j}{2^N})^q\} \quad (4.25)$$

where $\lambda_1 = 1+c_3+c_1+(-1)^{N/2}c_2$, $\lambda_2 = 1+c_3-c_1-(-1)^{N/2}c_2$, $\lambda_3 = 1-c_3+c_1-(-1)^{N/2}c_2$, $\lambda_4 = 1-c_3-c_1+(-1)^{N/2}c_2$.

Especially, if $c_1 = \alpha,\ c_2 = -\alpha, c_3 = 2\alpha-1$ and $N = 2$, then N-qubit states ρ is α-state. In FIG. 4.1, we display that the difference between the q-GQD of two α states (corresponding to $\alpha = 0.58$ and $\alpha = 0.3$, respectively), as a function q. This difference has negative or positive values by changing on the range of q. This is not indeed a common feature. This recovers the result in [23].

To conclude, we found an example of order relation which does not remain invariant of increasing or decreasing under a change of q in q-GQD.

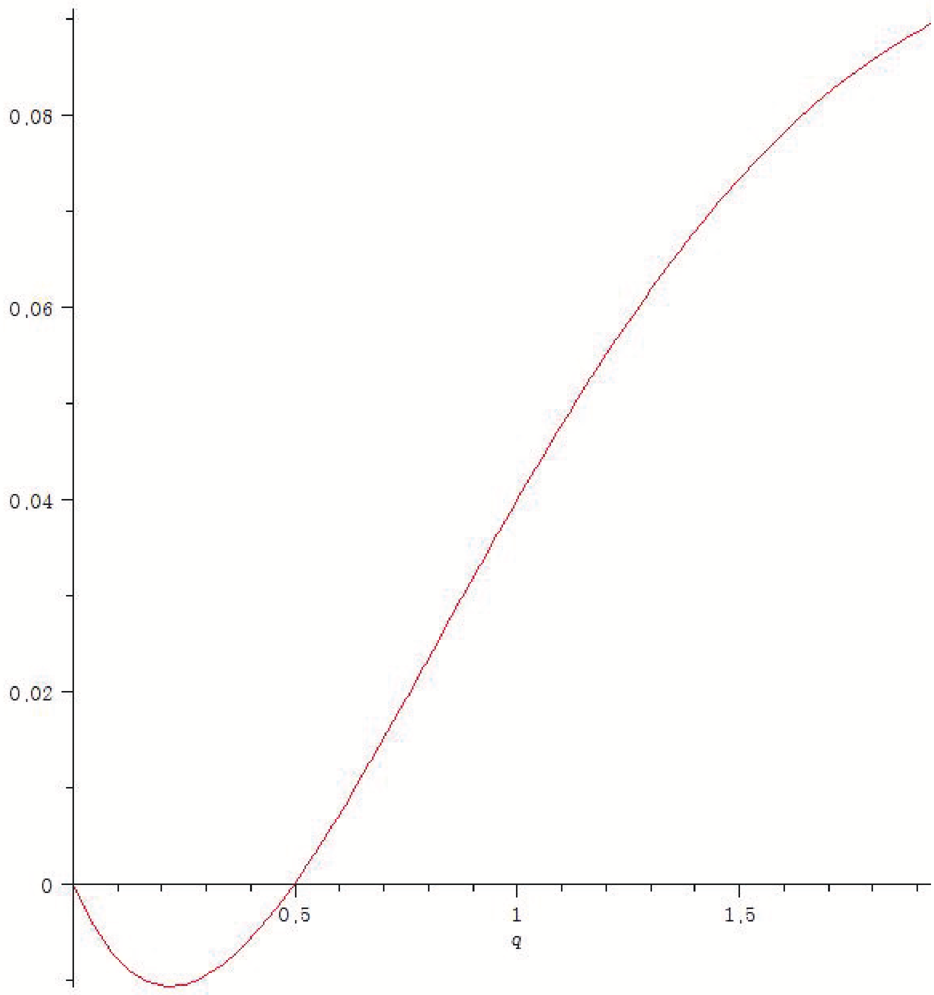

Figure 4.1: Difference between q-GQD of two α states as a function of the parameter q. A complicated ordering relation is discovered.

Let us introduce another property, which recovers the results in [25], of q-GQD. Let us begin by defining $GQD_q(\rho^{A_1\cdots A_N})$ in a quantum state $\rho^{A_1\cdots A_N}$ generated by a measurement $\Phi(\rho^{A_1\cdots A_N})$ is $GQD_q(A_1 : \cdots : A_N)_\Phi$. We can then show that $GQD_q(A_1 : \cdots : A_N)_\Phi$ can be decomposed in terms of loss of correlation for suitable bipartition, as provided by following theorem.

Theorem 4.2.4. *Given a non-selective measurement $\Phi(\rho^{A_1\cdots A_N})$, the loss of correlation $GQD_q(\rho^{A_1:\cdots:A_N})_\Phi$ can be decomposed as*

$$GQD_q(A_1 : \cdots : A_N)_\Phi = \sum_{k=1}^{N-1} GQD_q(A_1 \cdots A_k : A_{k+1})_\Phi \tag{4.26}$$

Proof.

$$\begin{aligned}
GQD_q(A_1 : \cdots : A_N)_\Phi &= \sum_{k=1}^{N}[S_q(\rho^{A_k}) - S_q(\Phi^{A_k}(\rho^{A_k}))] \\
&\qquad - S_q(\rho^{A_1\cdots A_N}) + S_q(\Phi(\rho^{A_1\cdots A_N})) \\
&= \sum_{k=1}^{N-1}[S_q(\rho^{A_k}) - S_q(\Phi^{A_k}(\rho^{A_k}))] \\
&\qquad - S_q(\rho^{A_1\cdots A_{N-1}}) + S_q(\Phi(\rho^{A_1\cdots A_{N-1}})) \\
&\qquad + S_q(\rho^{A_N}) - S_q(\Phi^{A_N}(\rho^{A_N})) + S_q(\rho^{A_1\cdots A_{N-1}}) \\
&\qquad - S_q(\Phi(\rho^{A_1\cdots A_{N-1}})) - S_q(\rho^{A_1\cdots A_N}) + S_q(\Phi(\rho^{A_1\cdots A_N})) \\
&= GQD_q(A_1 : \cdots : A_{N-1})_\Phi + GQD_q(A_1 \cdots A_{N-1} : A_N)_\Phi.
\end{aligned}$$

By recursively applying this process, we can reduce the term $GQD_q(A_1 \cdots A_{N-1} : A_N)_\Phi$ to a sum of bipartite contributions $\sum_{k=1}^{N-2} GQD_q(A_1 \cdots A_k : A_{k+1})_\Phi$. So, we can lead to the following equation:

$$GQD_q(A_1 : \cdots : A_N)_\Phi = \sum_{k=1}^{N-1} GQD_q(A_1 \cdots A_k : A_{k+1})_\Phi. \tag{4.27}$$

□

Theorem 4.2.5. *The bipartite* $GQD_q(AB : C)_\Phi$ *under a measurement* $\Phi(\rho^{ABC})$ *can not increase by discarding a subsystem, i.e.*

$$GQD_q(AB : C)_\Phi \quad = GQD_q(A : C)_\Phi$$

provided that the Tsallis conditional mutual information

$$\mathcal{I}_q(BC|A)_\rho \equiv S_q(\rho^{AB}) + S_q(\rho^{AC}) - S_q(\rho^{ABC}) - S_q(\rho^{A}) \tag{4.28}$$

does not increase after $\Phi(\rho^{ABC})$, *i.e.*

$$\mathcal{I}_q(BC|A) \geq \mathcal{I}_q(BC|A)_{\Phi(\rho)}. \tag{4.29}$$

Proof. By arranging the terms in this inequality $\mathcal{I}_q(BC|A) \geq \mathcal{I}_q(BC|A)_{\Phi(\rho)}$, we can obtain next inequality.

$$\begin{aligned} S_q(\rho^{AB}) - S_q(\rho^{ABC}) - S_q(\Phi(\rho^{AB})) + S_q(\Phi(\rho^{ABC})) \\ \geq S_q(\rho^{A}) - S_q(\rho^{AC}) - S_q(\Phi(\rho^{A})) + S_q(\Phi(\rho^{AC})). \end{aligned}$$

Namely, we lead to following equation:

$$GQD_q(AB : C)_\Phi \quad = GQD_q(A : C)_\Phi \tag{4.30}$$

with the constraint as

$$\mathcal{I}_q(BC|A) \geq \mathcal{I}_q(BC|A)_{\Phi(\rho)}. \tag{4.31}$$

□

Theorem 4.2.4 and 4.2.5 allow the relation between multipartite and pairwise correlation. By minimizing $GQD_q(A_1 : \cdots : A_N)_\Phi$ over all measurement $\Phi(\rho^{A_1\cdots A_N})$, we can obtain $GQD_q(\rho^{A_1\cdots A_N}) \equiv GQD_q(A_1 : \cdots : A_N)$. Then, we can easily get a monogamy bound for q-GQD in Theorem 4.2.6.

Theorem 4.2.6. *For an arbitrary density operator $\rho^{A_1\cdots A_N}$ with an amount of q-GQD in N-partite system given by $GQD_q(A_1 : \cdots : A_N)_\Phi$, pairwise q-GQD obeys the monogamy bound*

$$GQD_q(A_1 : \cdots : A_N) \quad \geq \sum_{k=1}^{N-1} GQD_q(A_1 : A_{k+1}) \tag{4.32}$$

provided that $\mathcal{I}_q(B_k A_{k+1}|A_1)_\rho \geq \mathcal{I}_q(B_k A_{k+1}|A_1)_{\Phi(\rho)}$, for $2 \leq k \leq N-1$, with $B_k = A_2 \cdots A_k$ and $\Phi(\rho)$ the minimizing measurement basis for $GQD_q(A_1 : \cdots : A_N)$.

Chapter 5

Conclusion

To sum up, we introduced the generalized global quantum discord, called the *q-global quantum discord* (q-GQD) which is defined in terms of the quantum Tsallis entropy and provided an equivalent expression for q-GQD. Furthermore we obtained the analytical expressions of q-GQD for the special classes of multi-qubit states and also proved the nonnegativity of q-GQD for the classes. In particular we found that this q-GQD has an unusual feature related to order and also shown a monogamy inequality for pairwise quantum correlations which implies that the sum of pairwise quantum correlations is upper limited by the multipartite quantum correlation with respect to q-GQD.

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국문 초록

살리스 엔트로피로 일반화 한 글로벌 양자 discord

이 경 진
서울대학원 수리과학부

이 학위논문에서 본 연구자는 살리스 엔트로피로 정의한 글로벌 양자 discord(q-GQD)를 제안한다. 그리고 두 클래스의 multipartite 상태에 대한 q-GQD 해석적 표현을 제시하고 q-GQD가 항상 0 이상임을 보인다. 또한 본 연구자는 이 q-GQD 가 q에 따른 대소관계에 대한 특별한 성질을 갖고 있는지 연구할 것이다. 또한 pairwise 양자 correlation 에 대한 모노가미 부등식을 증명할 것이다.

주요어: quantum discord, Global quantum discord, Tsallis entropy, monogamy
학 번: 2005–30965

감사의 글

제일 먼저 저에게 많은 가르침과 많은 격려를 해 주신 지동표 교수님과 김도한 교수님께 감사의 말씀을 드립니다. 언제나 너그러운 마음으로 이해해 주시고 배려해주신 교수님의 은혜 평생 잊지 못할 것 같습니다. 그리고 저에가 많은 가르침과 도움을 주신 이수준 교수님과 김정산 교수님께도 진심으로 감사의 말씀을 드립니다. 또한 바쁘신 와중에도 저의 논문을 심사해 주신 천정희 교수님, Yasuhiro Takahashi 박사님께도 큰 감사 드립니다.

양자 정보 및 양자 전산 연구실에서 가족과 같이 동고동락하며 지내왔고 많은 추억을 갖게 해준 정운이형, 갑균이형에게도 감사의 마음을 전합니다. 또한 연구실 책임자로서 마지막까지 저를 지원해 주신 태완이형에게도 큰 고마움을 전합니다.

고등학교 친구들인 동성, 재성, 이성, 성수, 동중, 대규, 진호 대학교 동기 효종이 힘들 때 마다 의지할 수 있었던 친구들이 존재하였기에 내가 포기하지 않고 끝까지 공부할 수 있었다. 이 자리를 빌어 진심으로 고맙다는 말 전하고 싶다. 그리고, 중동고등학교 교장선생님, 교감선생님, 수학과 선생님들과 입사 동기 지혁샘, 양현샘, 현우샘, 종목샘, 지연샘 그리고 중동고에 재직하시는 모든 선생님께도 감사의 마음을 전합니다. 고등학교에 재직하면서 학위 논문을 준비하기란 부족함이 많은 저에게 많은 어려움이 있었지만 선생님들의 도움으로 마무리할 수 있었습니다. 진심으로 감사드립니다. 그리고 2011학년도 1학년 6반 제자들과 2012학년도 1학년 12반 제자들과는 이 기쁨을 나누고 싶습니다.

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